PHYS 798C Fall 2025
Lecture 22 Summary

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I. JOSEPHSON JUNCTION CRITICAL CURRENT MODULATION WITH A MAGNETIC FIELD

Consider an SIS Josephson junction with barrier thickness $2a$ and superconducting electrodes with thickness much larger than the effective penetration depth. The middle of the barrier lies in the yz plane. Apply a dc magnetic field through the barrier parallel to the electrodes $\vec{B} = B\widehat{y}$.
By drawing a contour of width dz in the z-direction and height much greater than the effective penetration depths in the two electrodes, one can follow the evolution of the phase of the macroscopic quantum wavefunction on the contour and deduce a differential equation for the evolution of the gauge-invariant phase difference γ along the junction in the z-direction:
$\frac{\partial \gamma}{\partial z} = \frac{2\pi d_{eff}}{\Phi_0} B$
Here $d_{eff} = 2a + \lambda_1 + \lambda_2$ is the "magnetic thickness" of the barrier. This was derived under the assumption that the junction is "short" (width $L < \lambda_J$, where the Josephson penetration depth λ_J is derived below), and makes no significant alteration to the applied magnetic field through screening.
Integrating this result along the width of the junction yields,
$\gamma(z) = \gamma(0) + 2\pi \frac{d_{eff} LB}{\Phi_0} \frac{z}{L}$. This expression contains the total flux through the junction $\Phi_J = B d_{eff} L$ divided by the flux quantum,
$\gamma(z) = \gamma(0) + 2\pi \frac{\Phi_J}{\Phi_0} \frac{z}{L}$.

Integrating the current density over the area of the junction yields the total current through the junction,
$I = I_c \sin(\gamma(0)) \frac{\sin(\pi \Phi_J/\Phi_0)}{\pi \Phi_J/\Phi_0}$, where $I_c \equiv J_c W L$. This is the famous magnetic diffraction curve for modulation of the critical current of a JJ with external in-plane magnetic field.

At $\Phi_J = 0$ the current $J = J_c \sin(\gamma(z))$ is uniform over the junction and it has a maximum critical current. At $\Phi_J/\Phi_0 = 1$ there is a linear increase of γ from $\gamma(0)$ to $\gamma(0) + 2\pi$ from one edge of the junction to the other, creating a single-period sinusoidal oscillation of current through the junction. This results in zero net current through the junction. This condition creates a self-sustaining excitaiton in the junction known as a Josephson vortex. It resembles an Abrikosov vortex, but its core is in the Josephson junction, and thus it is pinned there and can only slide along the junction if acted upon by an external DC or RF current. The microwave properties of these Josephson vortices are studied by Sheng-Chiang Lee and Dan Oates.

The resemblance of the critical current diffraction pattern to single-slit diffraction in wave optics shows the analogy to interference created by spatial variation of a phase, $\gamma(z)$ in this case. In optics, one can imagine a sequence of Huygens point sources spanning the slit and radiating coherently in all directions. The superposition of all that radiation creates the famous Fraunhofer diffraction pattern on a screen beyond the slit. Similarly in the Josephson junction, each location has a unique value of the gauge-invariant phase difference $\gamma(z)$ and acts as an independent Josephson junction. The superposition of all these elementary Josephson junctions then adds up to the net current I flowing through the entire junction.

The "long junction" case considers the effect of self-generated modifications of the applied magnetic field due to screening. Starting with the differential equation derived above, $\frac{\partial \gamma}{\partial z} = \frac{2\pi d_{eff}}{\Phi_0} B$, now assume that $B = B_{ext} + B_{self}$ and that the total magnetic field must satisfy Ampere's law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon \mu_0 \frac{\partial \vec{E}}{\partial t}$. Assuming a static situation and that the edges of the junction are far away so that $\vec{B} = B_y(z)\widehat{y}$ only, then Ampere's law becomes $\frac{\partial B_y(z)}{\partial z} = -\mu_0 J_x(z)$. Combining this with the differential equation for $\gamma(z)$ yields the (time-independent) 1D sine-Gordon equation,
$\frac{d^2\gamma}{dz^2} = \frac{1}{\lambda_J^2} \sin(\gamma(z))$,
where $\frac{1}{\lambda_J^2} = \frac{2\pi \mu_0 d_{eff} J_c}{\Phi_0}$, which defines the Josephson penetration depth $\lambda_J \equiv \sqrt{\frac{\Phi_0}{2\pi \mu_0 d_{eff} J_c}}$. Note that the Josephson penetration depth is typically much larger than the London penetration depth at the same

T/T_c. This makes Josephson vortices very elongated in the junction direction, as shown in scanning SQUID microscope images on the class web site.

II. THE RSJ MODEL OF A JOSEPHSON JUNCTION

In general both quasiparticles and Cooper pairs can tunnel through the barrier in a Josephson junction when a potential difference (or equivalently $d\gamma/dt \neq 0$) is present. To include this possibility we treat the circuit model of a JJ as a parallel combination of an ideal Josephson junction (that obeys the two Josephson equations) and a resistor (that obeys a generalization of Ohm's law for nonlinear resistors). The resistance will in general depend on bias voltage and temperature, $R_N = R(V,T)$. This is known as the resistively shunted junction model (RSJ).
A bias current on the JJ will in general split between the two branches and produce a total current of, $I = I_c \sin\gamma + \frac{\Phi_0}{2\pi R_N}\frac{d\gamma}{dt}$.
Including the shunt capacitance of the junction adds a displacement current, and is described by the RCSJ model.